

Algebraic Geometry of Error Amplification: the Prony leaves

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Abstract

We provide an overview of some results of [1–4] on the “geometry of error amplification” in solving Prony system, in situations where the nodes near-collide. It turns out to be governed by the “Prony foliations” S_q , whose leaves are “equi-moment surfaces” in the parameter space. Next, we prove some new results concerning explicit parametrization of the Prony leaves.

1 Introduction

We consider the problem of the “measurements error amplification” in solving classical *Prony system* of algebraic equations, with the unknowns

a_j, x_j , $j = 1, \dots, d$, and with the right hand side formed by the known “noisy” measurements μ_0, \dots, μ_{2d-1} . This system has a form

$$\sum_{j=1}^d a_j x_j^k = \mu_k, \quad k = 0, 1, \dots, 2d-1. \quad (1.1)$$

We denote by $A = (a_1, \dots, a_d) \in \mathbb{R}^d$ and $X = (x_1, \dots, x_d) \in \mathbb{R}^d$ the unknowns in system (1.1), and denote by \mathcal{P}_d the “parameter space” of the unknowns (A, X) . We will always assume that the nodes X are pairwise different and ordered: $x_1 < x_2 < \dots < x_d$.

Prony system appears in many theoretical and applied mathematical problems. There exists a vast literature on Prony and similar systems - see, as a very small sample, [6, 8–10, 12, 25–29] and references therein. In particular, the bibliography in [6] contains more than 50 pages.

Some applications of Prony system are of major practical importance, and, in case when some of the nodes x_j nearly collide, it is well known to present major mathematical difficulties, in particular, in the context of “super-resolution problem” (see [1–4, 7, 8, 13–18, 23, 25] as a small sample).

The present paper deals with the problem of “error amplification” in solving a Prony system in the case that the nodes x_1, \dots, x_d nearly collide. Our approach is independent of a specific method of inversion and deals with a possible amplification of the measurements errors, in the reconstruction process, caused by the geometric nature of the Prony system.

The paper consists of two parts: the first (Section 3) is a summary of some recent results of [1–4] on the error amplification for near-colliding nodes. The main observation here is that the incorrect reconstructions, caused by the measurements noise, *are spread along certain algebraic subvarieties S_q in the parameter space, which we call the “Prony leaves”*.

Definition 1.1 *For $q = 0, \dots, 2d-1$, and $\mu = (\mu_0, \dots, \mu_q)$, the Prony leaf $S_q = S_q(\mu)$ is an algebraic variety in the parameter space \mathcal{P}_d , defined by the first $q+1$ equations of the Prony system (1.1):*

$$\sum_{j=1}^d a_j x_j^k = \mu_k, \quad k = 0, 1, \dots, q. \quad (1.2)$$

Generically, the dimension of the leaf $S_q(\mu)$ is $2d - q - 1$. The chain

$$S_0 \supset S_1 \supset \dots \supset S_{2d-2} \supset S_{2d-1}$$

can be explicitly computed (in principle), from the known measurements $\mu = (\mu_0, \dots, \mu_{2d-1})$. Notice that S_{2d-1} coincides with the set of solutions of the “full” Prony system (1.1).

In our approach the Prony leaves S_q serve as an approximation to the set of possible “noisy solutions” of (1.1) which appear for a noisy right-hand side μ . The Prony curve S_{2d-2} is especially prominent in the presentation below.

An important fact, found in [4], is that *if the nodes x_1, \dots, x_d form a cluster of a size $h \ll 1$, while the measurements error is of order ϵ , then the worst case error in reconstruction of S_q is of order ϵh^{-q} . Thus, for smaller q , the leaves S_q become bigger, but the accuracy of their reconstruction becomes better. The same is true for the accuracy with which S_q approximate noisy solutions of (1.1).* Compare Theorems 3.6 and 3.4 below.

In particular, the worst case error in reconstruction of the solution S_{2d-1} of (1.1) is $\sim \epsilon h^{-2d+1}$, while the worst case error in reconstruction of the Prony curve S_{2d-2} is of order ϵh^{-2d+2} . That is, the reconstruction of the Prony curve S_{2d-2} is h times better than the reconstruction of the solutions themselves.

Consequently, we can split the solution of (1.1) into two steps: first finding, with an improved accuracy, the Prony curve $S_{2d-2}(\mu)$, and then localizing *on this curve* the solution of (1.1). In particular, in the presence of a certain additional a priori information on the expected solutions of the Prony system (for example, upper and/or lower bounds on the amplitudes), it was shown in [4] that the Prony curves can be used in order to significantly improve the overall reconstruction accuracy.

We believe that the results of [1–4] presented in Section 3 justify a detailed algebraic-geometric study of the Prony leaves. In the second part of the present paper (Section 4) we prove some new results providing explicit equations, and explicit parametric representation, of the Prony leaves and their projections into the nodes space.

Finally, in Section 5, we summarize some open questions, naturally arising in the study of the error amplification and of the Prony leaves.

2 Setting of the problem

In this paper we adopt one of many equivalent settings for the problem of inversion of the Prony system. It is the problem of moment reconstruction

of spike-trains, that is, of linear combinations of d shifted δ -functions:

$$F(x) = \sum_{j=1}^d a_j \delta(x - x_j), \quad (2.1)$$

with $A = (a_1, \dots, a_d) \in \mathbb{R}^d$, $X = (x_1, \dots, x_d) \in \mathbb{R}^d$, $x_1 < x_2 < \dots < x_d$. We will consider signal (2.1) as the point (A, X) in the parameter space \mathcal{P}_d introduced above.

We assume that the form (2.1) of signals F is a priori known, but the specific parameters $(A, X) \in \mathcal{P}_d$ are unknown. Our goal is to reconstruct (A, X) from $2d$ moments $m_k(F) = \int_{-\infty}^{\infty} x^k F(x) dx$, $k = 0, \dots, 2d - 1$, which are known with a possible error bounded by $\epsilon > 0$.

An immediate computation shows that the moments $m_k(F)$ are expressed through the unknown parameters (A, X) as $m_k(F) = \sum_{j=1}^d a_j x_j^k$. Hence our reconstruction problem is equivalent to solving the Prony system (1.1), with $\mu_k = m_k(F)$.

Let a signal $F(x) \in \mathcal{P}_d$ be fixed. In order to describe the geometry of the error amplification in solving (1.1) we define, following [3, 4], the ϵ -error set $E_\epsilon(F) \subset \mathcal{P}_d$. It consists of all signals $F'(x) \in \mathcal{P}_d$ which may appear in the reconstructions of F from noisy moment measurements μ'_k , $|\mu'_k - m_k(F)| \leq \epsilon$, $k = 0, \dots, 2d - 1$. Formally we have the following definition:

Definition 2.1 *The error set $E_\epsilon(F) \subset \mathcal{P}_d$ is the set consisting of all the signals $F'(x) \in \mathcal{P}_d$ with*

$$|m_k(F') - m_k(F)| \leq \epsilon, \quad k = 0, \dots, 2d - 1. \quad (2.2)$$

Our ultimate goal is a detailed understanding of the geometry of the error set $E_\epsilon(F)$, in the cases where the nodes of F near-collide, and applying this information in order to improve the reconstruction accuracy.

We can explicitly describe the ϵ -error set $E_\epsilon(F)$, considering the moments $m_k = m_k(F')$, $k = 0, \dots, 2d - 1$, as non-linear coordinates in the space \mathcal{P}_d of signals F' . Indeed, inequalities (2.2) immediately show that *in these coordinates the error set $E_\epsilon(F)$ is the coordinate ϵ -cube in \mathcal{P}_d , centered at F .*

However, there are serious difficulties with this description. First, at the points where the nodes collide, the moment coordinates develop complicated singularities. In particular, they fail to form a coordinate system near these

points. Still, the description of the error set $E_\epsilon(F)$ via algebraic inequalities (2.2) remains valid.

Secondly, in the case of the nodes X forming a cluster of size $h \ll 1$, the “moment coordinate system” turns out to be significantly “stretched” in some directions, up to the order $(\frac{1}{h})^{2d-1}$. Therefore the description of $E_\epsilon(F)$ in the “moment coordinate system” given by (2.2) requires a “translation” into the natural coordinates (A, X) in the space \mathcal{P}_d . This translation (in a certain neighborhood of F in \mathcal{P}_d) is the main result of [4] (see a review in Section 3 below).

We can aim to understand *the global geometry* of $E_\epsilon(F)$ via an algebraic-geometric investigation of the moment coordinates, and, in particular, of the Prony leaves. In the case of two nodes, this investigation was started in [3, 4]. In Section 4 below we extend the results of [3] to a general case of d nodes.

3 Summary of the results in [1–4]

Let a signal $F \in \mathcal{P}_d$ be given. We denote $I_F = [x_1, x_d]$ the minimal interval in \mathbb{R} containing all the nodes x_1, \dots, x_d . We put $h(F) = \frac{1}{2}(x_d - x_1)$ to be the half of the length of I_F , and put $\kappa(F) = \frac{1}{2}(x_1 + x_d)$ to be the central point of I_F .

We “normalize” the signal F , shifting the interval I_F to have its center at the origin, and then rescaling I_F to the interval $[-1, 1]$. For this purpose we consider, for each $\kappa \in \mathbb{R}$ and $h > 0$ the transformation

$$\Psi_{\kappa, h} : \mathcal{P}_d \rightarrow \mathcal{P}_d, \quad (3.1)$$

defined by $(A, X) \rightarrow (A, \tilde{X})$, with

$$\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_d), \quad \tilde{x}_j = \frac{1}{h}(x_j - \kappa), \quad j = 1, \dots, d.$$

For a given signal F we put $h = h(F)$, $\kappa = \kappa(F)$ and call the signal $G = \Psi_{\kappa, h}(F)$ the model signal for F . Clearly, $h(G) = 1$ and $\kappa(G) = 0$. Explicitly G is written as

$$G(x) = \sum_{j=1}^d a_j \delta(x - \tilde{x}_j).$$

With a certain misuse of notations, we will denote the space \mathcal{P}_d containing the model signals G by $\tilde{\mathcal{P}}_d$, and call it “the model space”. For $F \in \mathcal{P}_d$ and $G = \Psi_{\kappa,h}(F)$, the moments of G

$$\tilde{m}_k(F) = m_k(G) = \sum_{j=1}^d a_j \tilde{x}_j^k, \quad k = 0, 1, \dots \quad (3.2)$$

are called the model moments of F .

Below we describe the error set of F in the associated model space $\tilde{\mathcal{P}}_d$, and use the associated model moments.

The main reason for mapping a general signal F into the model space is that in the case of nodes X , forming a cluster of size $h \ll 1$, as it was mentioned above, the moment coordinates turn out to be “stretched” in some directions, up to the order $(\frac{1}{h})^{2d-1}$. *In contrast, in the model space $\tilde{\mathcal{P}}_d$ the system m_0, \dots, m_{2d-1} is compatible with the standard coordinates (A, \tilde{X}) of $\tilde{\mathcal{P}}_d$, for all signals G with “well-separated nodes” (see Theorem 3.3 below).*

For a given $F \in \mathcal{P}_d$, with the model signal $G = \Psi_{\kappa,h}(F)$, we denote by $\tilde{E}_\epsilon(F)$ the set $\Psi_{\kappa,h}(E_\epsilon(F))$, which represents the error set $E_\epsilon(F)$ of F in the model space $\tilde{\mathcal{P}}_d$. Note that $\tilde{E}_\epsilon(F)$ is simply a translated and rescaled version of $E_\epsilon(F)$

For given $\epsilon, h > 0$ denote by $\Pi_{\epsilon,h}(G)$ the “curvilinear parallelepiped” consisting of all $G' \in \tilde{\mathcal{P}}_d$ satisfying the inequalities

$$|m_k(G') - m_k(G)| \leq \epsilon h^{-k}, \quad k = 0, \dots, 2d-1.$$

Theorem 3.1 *For any $F \in \mathcal{P}_d$, let $\kappa = \kappa(F)$ and $h = h(F)$. Let $G = \Psi_{\kappa,h}(F)$ be the model signal for F . Then, for any $\epsilon > 0$ we have*

$$\Pi_{\epsilon'',h}(G) \subset \tilde{E}_\epsilon(F) \subset \Pi_{\epsilon',h}(G),$$

where $\epsilon' = (1 + |\kappa|)^{2d-1}\epsilon$, $\epsilon'' = (1 + |\kappa|)^{-2d+1}\epsilon$. Specifically, for $\kappa = \kappa(F) = 0$,

$$\tilde{E}_\epsilon(F) = \Pi_{\epsilon,h}(G).$$

The result of Theorem 3.1 holds without any assumptions on the mutual relation of ϵ and h , or on the distances between the nodes of F . However, without such assumptions it is difficult to provide any specific geometric information on the moment parallelepiped $\Pi_{\epsilon,h}(G)$. Still, Theorem 3.1 implies

an important fact: *the Prony leaves S_q of the model signal G globally form a “skeleton” of the error set $\tilde{E}_\epsilon(F)$, and, in case when ϵ and h tend to zero in a certain rate, S_q are the limits of $\tilde{E}_\epsilon(F)$.*

To formulate this result accurately, let us denote the Prony leaves S_q passing through G by $S_q(G)$. Thus $S_q(G) = S_q(\mu)$ for $\mu = (\mu_0, \dots, \mu_q)$ with $\mu_k = m_k(G)$, $k = 0, \dots, q$.

Now let us assume that a model signal

$$G = \sum_{j=1}^d a_j \delta(x - \tilde{x}_j) \in \tilde{\mathcal{P}}_d$$

is fixed. For each real κ and $h > 0$ we consider a signal $F_{\kappa,h}(G)$, obtained from G by an h -scaling and κ -shift of x :

$$F_{\kappa,h}(G) = \Psi_{\kappa,h}^{-1}(G) = \sum_{j=1}^d a_j \delta(x - x_j) \in \mathcal{P}_d, \quad x_j = h\tilde{x}_j + \kappa.$$

Thus $G = \Psi_{\kappa,h}(F_{\kappa,h}(G))$ remains the model signal for each $F_{\kappa,h}(G)$.

For each $0 \leq q \leq 2d - 1$ and $c > 0$ denote by $S_q(G, c)$ the part of the Prony leaf $S_q(G)$, consisting of $G' \in S_q(G)$ with $|m_{q+1}(G') - m_{q+1}(G)| \leq c$.

Theorem 3.2 *Let $0 \leq q \leq 2d - 1$, κ and $C > 0$ be fixed. Then for $h \rightarrow 0$ and for $\epsilon = Ch^{q+1}$, the error set $\tilde{E}_\epsilon(F_{\kappa,h}(G)) = \Psi_{\kappa,h}(E_\epsilon(F_{\kappa,h}(G)))$ converges to the part $\bar{S}_q(G)$ of the Prony leaf $S_q(G)$, satisfying*

$$S_q(G, c'') \subset \bar{S}_q(G) \subset S_q(G, c'), \quad c' = C(1 + |\kappa|)^{2d-1}, \quad c'' = C(1 + |\kappa|)^{-2d+1}.$$

Figures 1 and 2 illustrate the case $d = 2, q = 2d - 2 = 2$ of Theorem 3.2.

This theorem shows that the Prony leaves $S_q(G)$ globally approximate the error set $\tilde{E}_\epsilon(F)$, for $h \ll 1$ and $\epsilon \sim h^{q+1}$. We consider the study of the “before limit” accuracy of this approximation as an important open question, which, presumably, can be treated with the tools of real algebraic geometry. Some initial results in this direction, obtained in [4], and based on a “quantitative” version of the inverse function theorem, are presented below.

In order to apply this theorem, we have to make explicit assumptions on the separation of the nodes X of our signal F , and on the size of its amplitudes A :

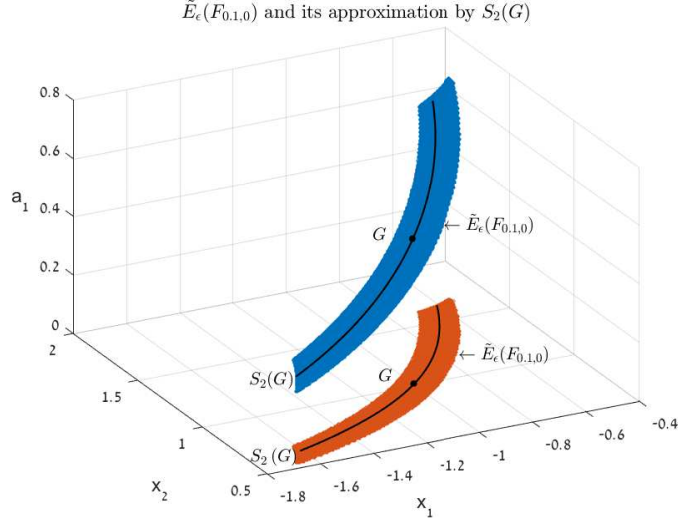


Figure 1: Presented are projections of the error set $\tilde{E}_\epsilon(F_{h,\kappa})$ and a section of the Prony curve $S_2(G)$, for $G = \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$, $h = 0.1$, $\kappa = 0$ and $\epsilon = h^3$. Stretched upwards is the projection into the coordinate subspace of x_1, x_2, a_1 and on the bottom plane into the nodes subspace x_1, x_2 .

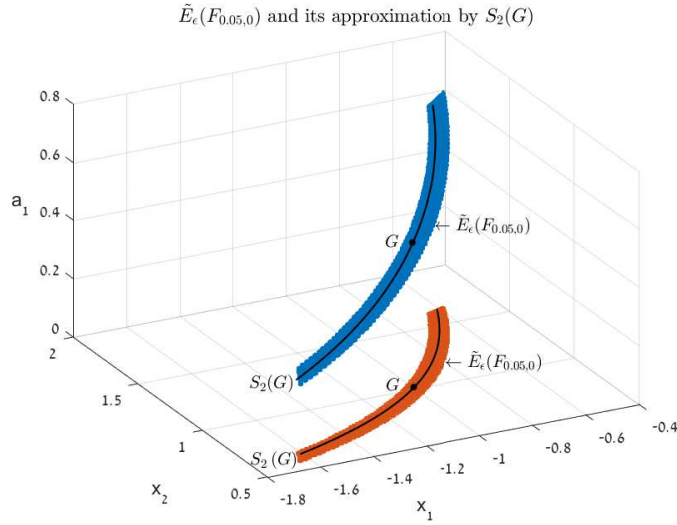


Figure 2: Presented are the error set $\tilde{E}_\epsilon(F_{h,\kappa})$ and a section of $S_2(G)$ for $G = \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$, $h = 0.05$, $\kappa = 0$ and $\epsilon = h^3$. Note the convergence of $\tilde{E}_\epsilon(F_{h,\kappa})$ to $S_2(G)$.

Definition 3.1 Let η satisfying $0 < \eta \leq \frac{2}{d-1}$, $d > 1$, and m, M with $0 < m < M$, be given. A signal $G \in \tilde{\mathcal{P}}_d$ is called (η, m, M) -regular if for each $j = 1, \dots, d-1$ the distance between the neighbor nodes $\tilde{x}_j, \tilde{x}_{j+1}$ of G is at least η , and the amplitudes a_1, \dots, a_d satisfy $m \leq |a_j| \leq M$, $j = 1, \dots, d$.

We say that a signal $F \in \mathcal{P}$ is (h, κ, η, m, M) -regular, if it can be obtained from an (η, m, M) -regular signal G by an h -scaling, and then a shift by κ .

We want to show that for an (η, m, M) -regular signal $G \in \tilde{\mathcal{P}}_d$ the model moments m_0, \dots, m_{2d-1} indeed form a coordinate system near G , which agrees with the standard coordinates A, \tilde{X} on $\tilde{\mathcal{P}}_d$:

Definition 3.2 The moment metric $d(G', G'')$ on $\tilde{\mathcal{P}}_d$ is defined through the model moments m_0, \dots, m_{2d-1} as

$$d(G', G'') = \max_{k=0}^{2d-1} |m_k(G'') - m_k(G')|.$$

Theorem 3.3 Let $G \in \tilde{\mathcal{P}}_d$ be an (η, m, M) regular signal. Then there are constants R, C_1, C_2 , depending only on d, η, m, M , such that:

1. The model moments $m_k = m_k(G')$ form a regular analytic coordinate system on the ball $B_R(G)$, centered at G , of radius R in the Euclidean metric on $\tilde{\mathcal{P}}_d$.
2. The moment metric $d(G', G'')$ is Lipschitz equivalent on $B_R(G)$ to the Euclidean metric $\|G'' - G'\|$: for each $G', G'' \in B_R(G)$ we have

$$C_1 d(G', G'') \leq \|G'' - G'\| \leq C_2 d(G', G'').$$

Consequently, our description of the error set in terms of the model moments, given in Theorems 3.1 and 3.2 above, can be translated, inside the ball $B_R(G)$, into a description in the standard coordinates:

Theorem 3.4 Let F be an (h, κ, η, m, M) -regular signal, and let $G \in \tilde{\mathcal{P}}_d$ be the model signal of F . Then for each $q = 0, \dots, 2d-1$ the “local” error set $\tilde{E}_\epsilon(F) \cap B_R(G)$ is contained in the Δ_q -neighborhood (in the Euclidean metric) of the Prony leaf $S_q(G)$, for

$$\Delta_q = C_2 \left(\frac{1 + |\kappa|}{h} \right)^q \epsilon.$$

Thus, as it was stated in the introduction, for smaller q , the leaves $S_q(G)$ become bigger, but the accuracy with which they approximate the error set $\tilde{E}_\epsilon(G)$ becomes better. The same is true for the accuracy of the reconstruction of the leaves $S_q(G)$ from the noisy measurements: compare with Theorem 3.6 below.

3.1 Worst case reconstruction error

We now present lower and upper bounds for the worst case reconstruction error $\rho(F, \epsilon)$, defined by

$$\rho(F, \epsilon) = \max_{F' \in E_\epsilon(F)} \|F' - F\|.$$

In a similar way we define $\rho^A(F, \epsilon)$ and $\rho^X(F, \epsilon)$ - the worst case errors in reconstruction of the amplitudes $A = (a_1, \dots, a_d)$ and of the nodes $X = (x_1, \dots, x_d)$ of F : considering signals F' with parameters A', X' we put

$$\rho^A(F, \epsilon) = \max_{F' \in E_\epsilon(F)} \|A' - A\|, \quad \rho^X(F, \epsilon) = \max_{F' \in E_\epsilon(F)} \|X' - X\|.$$

Theorem 3.5 *Let $F \in \mathcal{P}_d$ be an (h, κ, η, m, M) -regular signal. Then there are constants C_3, K_1, K_2, K_3, K_4 depending only on d, κ, η, m, M , such that for each positive $\epsilon \leq C_3 h^{2d-1}$ the following bounds for the worst case reconstruction errors are valid:*

$$K_1 \epsilon h^{-2d+1} \leq \rho(F, \epsilon), \quad \rho^A(F, \epsilon) \leq K_2 \epsilon h^{-2d+1},$$

$$K_3 \epsilon h^{-2d+2} \leq \rho^X(F, \epsilon) \leq K_4 \epsilon h^{-2d+2}.$$

Theorem 3.5 shows that the noise level $\epsilon = \epsilon_0 = C_3 h^{2d-1}$ plays a role of a threshold in noisy reconstruction: for $\epsilon < \epsilon_0$ the worst case error remain bounded and decreases with ϵ . However, for $\epsilon \sim \epsilon_0$ and bigger the nodes may collide and the amplitudes may blow up to infinity.

For $F \in \mathcal{P}_d$ and G the model signal of F , we define the worst case reconstruction error of the Prony leaves $S_q(G)$, in the model space $\tilde{\mathcal{P}}_d$, as follows. For $q = 0, \dots, 2d-1$

$$\rho^{S_q}(F, \epsilon) = \max_{G' \in \tilde{E}_\epsilon(F)} d_H(S_q(G) \cap B_R(G), S_q(G') \cap B_R(G)).$$

Here $d_H(W_1, W_2)$ is the Hausdorff distance between the sets W_1, W_2 , and $B_R(G) \subset \tilde{\mathcal{P}}_d$ is the ball defined in Theorem 3.3.

Theorem 3.6 *Let $F \in \mathcal{P}_d$ be an (h, κ, η, m, M) -regular signal, and let G be the model of F . Then, for each positive $\epsilon \leq C_3 h^{2d-1}$ the following bounds for the worst case reconstruction errors of the Prony leaves $S_q(G)$, $q = 0, \dots, 2d-1$, are valid:*

$$K_1 \epsilon h^{-q} \leq \rho^{S_q}(F, \epsilon) \leq K_2 \epsilon h^{-q}.$$

4 Explicit parametrization of Prony leaves

In this section we show that the Prony leaves $S_q(\mu)$ allow for an explicit parametrization. For $q \leq d-1$ (Section 4.1) this parametrization is produced in a rather straightforward way, via expressing some of the amplitudes a_j through the nodes and the remaining amplitudes. It requires solving linear systems with Vandermonde matrices on the nodes x_j . For $q \geq d$ (Section 4.2) our parametrization is produced via a proper modification of the classical solution method of the Prony system (suggested, essentially, already in [29]). It requires solving linear systems with Hankel matrices on the moments μ_k , and subsequently finding the roots of a univariate polynomial. In Section 4.3 we illustrate the results of Sections 4.1 and 4.2, providing a complete description of the Prony leaves in the case of two nodes.

4.1 Prony leaves S_q for $q \leq d-1$

As above, we consider signals $F = (A, X) = \sum_{j=1}^d a_j \delta(x - x_j) \in \mathcal{P}_d$. We denote by \mathcal{P}_d^A and \mathcal{P}_d^X the spaces of the amplitudes $A = (a_1, \dots, a_d)$ and of the nodes $X = (x_1, \dots, x_d)$, respectively, and denote by π the projection

$$\pi : \mathcal{P}_d \cong \mathcal{P}_d^A \times \mathcal{P}_d^X \rightarrow \mathcal{P}_d^X.$$

For a given $\mu = (\mu_0, \dots, \mu_q)$ we consider the Prony leaf $S_q(\mu) \subset \mathcal{P}_d$,

Theorem 4.1 *For $q \leq d-1$ and for any $\mu = (\mu_0, \dots, \mu_q)$ the Prony leaf $S_q(\mu)$ is a smooth subvariety of \mathcal{P}_d of dimension $2d - q - 1$.*

The projection $\pi : S_q(\mu) \rightarrow \mathcal{P}_d^X$ is onto, and forms a regular locally trivial fibration over \mathcal{P}_d^X . The fibers of π are affine subvarieties of dimension $d - q - 1$ in \mathcal{P}_d .

The amplitudes a_{q+2}, \dots, a_d and all the nodes $x_1 < x_2 < \dots < x_d$ can be chosen as the coordinates on $S_q(\mu)$, while the amplitudes a_1, \dots, a_{q+1} are expressed in these coordinates as

$$a_j = \sum_{l=0}^q A_l(X) \mu_l + \sum_{s=q+2}^d B_s(X) a_s, \quad j = 1, \dots, q+1,$$

with $A_l(X), B_s(X)$ regular rational functions in $X = (x_1, \dots, x_d)$.

Proof:

In the coordinates $(A = (a_1, \dots, a_d), X = (x_1, \dots, x_d))$ in \mathcal{P}_d the Prony leaf $S_q(\mu)$ is defined by the following equations:

$$\begin{aligned} a_1 + a_2 + \dots + a_d &= \mu_0 \\ a_1 x_1 + a_2 x_2 + \dots + a_d x_d &= \mu_1 \\ a_1 x_1^2 + a_2 x_2^2 + \dots + a_d x_d^2 &= \mu_2 \\ &\dots\dots\dots \\ a_1 x_1^q + a_2 x_2^q + \dots + a_d x_d^q &= \mu_q \end{aligned} \tag{4.1}$$

We can rewrite equations (4.1) as

$$\begin{aligned} a_1 + a_2 + \dots + a_{q+1} &= \mu_0 - a_{q+2} - \dots - a_d \\ a_1 x_1 + a_2 x_2 + \dots + a_{q+1} x_{q+1} &= \mu_1 - a_{q+2} x_{q+2} - \dots - a_d x_d \\ a_1 x_1^2 + a_2 x_2^2 + \dots + a_{q+1} x_{q+1}^2 &= \mu_2 - a_{q+2} x_{q+2}^2 - \dots - a_d x_d^2 \\ &\dots\dots\dots \\ a_1 x_1^q + a_2 x_2^q + \dots + a_{q+1} x_{q+1}^q &= \mu_q - a_{q+2} x_{q+2}^q - \dots - a_d x_d^q \end{aligned} \tag{4.2}$$

The left hand side of (4.2) is the Vandermonde linear system with respect to a_1, \dots, a_{q+1} . Hence we can express from (4.2) the amplitudes a_1, \dots, a_{q+1} via the Cramer rule. The resulting expressions will be linear in μ and in a_{q+2}, \dots, a_d , with the coefficients - rational functions in the nodes. Notice that the denominator is the Vandermonde determinant $V_q(x_1, \dots, x_{q+1}) = \prod_{1 \leq i < j \leq q+1} (x_j - x_i)$.

For any fixed $X = (x_1, \dots, x_d)$ the fiber of π over X is an affine subset in \mathcal{P}_d parametrized by a_{q+2}, \dots, a_d . This completes the proof of Theorem 4.1.

□

Let us stress a special case $q = d - 1$. In this case we have

$$a_j = \frac{1}{V_d(x_1, \dots, x_d)} \sum_{l=0}^q A_l(X) \mu_l, \quad j = 1, \dots, d. \quad (4.3)$$

These expressions remain valid also on the Prony leaves S_q for $q \geq d$.

4.2 Prony leaves S_q for $q \geq d$

4.2.1 Projections $S_q^X(\mu)$ of $S_q(\mu)$ onto the nodes subspace

Starting with $q = d$ the dimension $2d - q - 1$ of the Prony leaves $S_q(\mu)$ is smaller than d . Consequently, the projections $S_q^X(\mu)$ of $S_q(\mu)$ onto the nodes subspace \mathcal{P}_d^X are proper subvarieties in \mathcal{P}_d^X . On the other hand, by (4.3), the amplitudes a_j on $S_q(\mu)$ can be uniquely reconstructed from the nodes X (and from μ). Accordingly, we first describe the equations, defining the projections $S_q^X(\mu)$ of $S_q(\mu)$ onto the nodes subspace \mathcal{P}_d^X . To obtain these equations we have to eliminate the amplitudes a_1, \dots, a_d from the equations (4.1). This can be achieved by substituting into (4.1) the expressions for a_j from (4.3). However, for $d > 2$ this leads to rather complicated expressions. Instead we use a modification of the classical solution method of the Prony system. Let

$$\begin{aligned} \sigma_1(x_1, \dots, x_d) &= -(x_1 + \dots + x_d) \\ \sigma_2(x_1, \dots, x_d) &= x_1x_2 + x_1x_3 + \dots + x_{d-1}x_d \\ &\dots\dots\dots \\ \sigma_d(x_1, \dots, x_d) &= (-1)^d x_1x_2 \cdot \dots \cdot x_{d-1}x_d \end{aligned} \quad (4.4)$$

be the Vieta elementary symmetric polynomials in x_1, \dots, x_d . We also put $\sigma_0 = 1$. Thus σ_j are the coefficients of the univariate polynomial

$$Q(z) = \prod_{j=1}^d (z - x_j) = z^d + \sigma_1 z^{d-1} + \dots + \sigma_d = \sum_{i=0}^d \sigma_{d-i} z^i,$$

whose roots are the nodes x_1, \dots, x_d .

The following system of $q - d + 1$ linear equations for $\sigma_1, \dots, \sigma_d$ forms a part of the standard (and classical) linear system for the coefficients of the polynomial Q (see, for instance, [25, 28, 29]. For $q = 2d - 1$ the complete system is obtained):

$$\begin{aligned}
\mu_{d-1}\sigma_1 + \mu_{d-2}\sigma_2 + \dots + \mu_0\sigma_d &= -\mu_d \\
\mu_d\sigma_1 + \mu_{d-1}\sigma_2 + \dots + \mu_1\sigma_d &= -\mu_{d+1} \\
&\dots\dots\dots \\
\mu_{q-1}\sigma_1 + \mu_{q-2}\sigma_2 + \dots + \mu_{q-d}\sigma_d &= -\mu_q
\end{aligned} \tag{4.5}$$

Taking into account that $\sigma_0 = 1$, this system can be rewritten as

$$\sum_{i=0}^d \mu_{l-i}\sigma_i = 0, \quad l = d, \dots, q.$$

System (4.5), being a linear system in variables $\sigma_1, \dots, \sigma_d$, forms a nonlinear system in x_1, \dots, x_d , if we consider σ_j as the Vieta elementary symmetric polynomials in x_1, \dots, x_d . We denote by $Y_q(\mu) \subset \mathcal{P}_d^X$ the variety of zeroes of this last system.

Theorem 4.2 *For $q \geq d$ the projection $S_q^X(\mu)$ of $S_q(\mu)$ onto the nodes subspace \mathcal{P}_d^X coincides with $Y_q(\mu)$.*

Proof: First we show that for $q \geq d$ system (4.1) implies system (4.5). Indeed, for each $l = d, \dots, q$ we obtain, using (4.1), that

$$\sum_{i=0}^d \mu_{l-i}\sigma_i = \sum_{i=0}^d \sigma_i \sum_{j=1}^d a_j x_j^{l-i} = \sum_{j=1}^d a_j \sum_{i=0}^d \sigma_i x_j^{l-i} = \sum_{j=1}^d a_j x_j^{l-d} Q(x_j) = 0,$$

since each node x_j is a root of $Q(x)$. In other words, for each $(A, X) \in \mathcal{P}_d$ satisfying system (4.1), the component X satisfies system (4.5). We conclude that the projection $S_q^X(\mu)$ of $S_q(\mu)$ onto the nodes subspace \mathcal{P}_d^X is contained in $Y_q(\mu)$.

To prove the opposite inclusion, let us assume that $X = (x_1, \dots, x_d) \in Y_q(\mu) \subset \mathcal{P}_d^X$, i.e. $X = (x_1, \dots, x_d)$ satisfies system (4.5). We uniquely define the amplitudes $A = (a_1, \dots, a_d)$ from the Vandermonde linear system, formed by the first d equations of system (4.1), according to expressions (4.3). Now we form a signal

$$F(x) = \sum_{j=1}^d a_j \delta(x - x_j) = (A, X) \in \mathcal{P}_d,$$

which by construction satisfies the first d equations of system (4.1). It remains to show that the last $q - d + 1$ equations of (4.1) are satisfied for $F(x)$.

Consider the rational function $R(z) = \sum_{j=1}^d \frac{a_j}{z-x_j}$. We have $R(z) = \frac{P(z)}{Q(z)}$ for a certain polynomial $P(z)$ of degree $d - 1$ and for

$$Q(z) = \prod_{j=1}^d (z - x_j) = z^d + \sigma_1 z^{d-1} + \dots + \sigma_d,$$

where $\sigma_i = \sigma_i(x_1, \dots, x_d)$, $i = 1, \dots, d$, are, as above, the Vieta elementary symmetric polynomials in x_1, \dots, x_d .

Developing the elementary fractions in $R(z)$ into geometric progressions, we get

$$R(z) = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}}, \quad m_k = m_k(F) = \sum_{j=1}^d a_j x_j^k. \quad (4.6)$$

Therefore, the moments $m_k = m_k(F)$, $k = 0, 1, \dots$, given by the left hand side $\sum_{j=1}^d a_j x_j^k$ of system (4.1), are the Taylor coefficients of the rational function $R(z) = \frac{P(z)}{Q(z)}$, with $P(z)$ of degree $d - 1$, and $Q(z)$ of degree d . Starting with $k = d$ these Taylor coefficients m_k of R are known to satisfy the recurrence relation

$$m_k = - \sum_{s=1}^d \sigma_s m_{k-s}, \quad (4.7)$$

σ_l being the coefficients of the denominator $Q(z)$ of $R(z)$. Since by the choice of the amplitudes a_j the first d equations of system (4.1) are satisfied, we conclude that $m_k = \mu_k$, $k = 0, \dots, d - 1$.

Now we use the assumption that system (4.5) is satisfied. Its equations show that μ_k satisfy exactly the same recurrence relation till $k = q$. Since the first d terms are the same, we conclude that in fact $m_k = \mu_k$, $k = 0, \dots, q$.

This means that the entire system (4.1) is satisfied. Consequently, $F \in S_q(\mu)$, and therefore $X = (x_1, \dots, x_d) \in S_q^X(\mu) \subset \mathcal{P}_d^X$. We conclude that $Y_q(\mu) \subset S_q^X$. This completes the proof of Theorem 4.2. \square

Remark In the above setting of Theorem 4.2 we do not make any assumption on the rank of linear system (4.5). It is easy to give examples of a right-hand

side $\mu = (\mu_0, \dots, \mu_q)$ of (4.5) for which the solutions of this system form an empty set, or an affine subspace $L_q(\mu)$ of any dimension not smaller than $2d - q - 1$. Theorem 4.2, as well as Theorem 4.3 below, remain true in each of this cases. Compare a detailed discussion of the situation for two nodes ($d = 2$) in Section 4.3 below.

The possible degenerations of system (4.5) are closely related to the conditions of solvability of Prony system (see, for example, Theorem 3.6 of [11]), and the discussion thereafter. Both these questions are very important in the robustness analysis of the Prony inversion, but we do not discuss them here.

4.2.2 Parametrization of Prony leaves S_q for $q \geq d$

Theorem 4.2 allows us to construct an explicit parametrization of the Prony leaves $S_q(\mu)$, $q \geq d$. It is enough to produce a parametrization of the projections $S_q^X(\mu)$, since the amplitudes a_j are expressed through the nodes x_j and μ via formulas (4.3).

Essentially, we follow the classical solution method of Prony systems, splitting it into two steps: first, solving a *linear system* (4.5) with respect to the variables σ_i , and then finding the roots of a *univariate polynomial* $Q(z)$ with the coefficients σ_i .

Let $V_d \cong \mathbb{R}^d$ be the space of the coefficients $\sigma = (\sigma_1, \dots, \sigma_d)$ of the polynomials $Q(z)$ (which we identify with the space of the polynomials Q themselves). Let $\mu = (\mu_0, \dots, \mu_q)$ be given. Equations (4.5) define an affine subspace $L_q(\mu) \subset V_d$, which is generically of dimension $2d - q - 1$ (but, depending on μ , $L_q(\mu)$ may be empty, or of any dimension not smaller than $2d - q - 1$).

Consider a subset $H_d \subset V_d$, consisting of *hyperbolic* polynomials Q , i.e. of those $Q(z) = z^d + \sigma_1 z^{d-1} + \dots + \sigma_d$ with all the roots real (and pairwise different - so we exclude the boundary). Hyperbolic polynomials correspond to some of the connected components of the complement in V_d of the discriminant set $\Delta_d \subset V_d$. The set H_d is important in many problems, and it was intensively studied (see, as a small sample, [5, 20] and references therein). We denote $L_q^h(\mu)$ the intersection of $L_q(\mu)$ and the set H_d of hyperbolic polynomials.

Definition 4.1 The “root mapping” $RM_d : H_d \rightarrow \mathcal{P}_d^X$ is defined by

$$RM_d(Q) = X = (x_1, \dots, x_d) \in \mathcal{P}_d^X,$$

where $x_1 < x_2 < \dots < x_d$ are the ordered roots of the hyperbolic polynomial $Q(z) \in H_d$.

The “Vieta mapping” $\mathcal{V}_d : \mathcal{P}_d^X \rightarrow H_d$ is defined by

$$\mathcal{V}_d(x_1, \dots, x_d) = (\sigma_1(x_1, \dots, x_d), \dots, \sigma_d(x_1, \dots, x_d)),$$

where $\sigma_i = \sigma_i(x_1, \dots, x_d)$, $i = 1, \dots, d$, are the Vieta elementary symmetric polynomials in x_1, \dots, x_d .

Clearly, on H_d the root mapping RM_d is regular, and $RM_d = \mathcal{V}_d^{-1}$. Therefore both the mappings $RM_d : H_d \rightarrow \mathcal{P}_d^X$ and its inverse $\mathcal{V}_d : \mathcal{P}_d^X \rightarrow H_d$ provide an isomorphism between H_d and \mathcal{P}_d^X .

Now we have all the tools required to describe the parametrization of the Prony leaves:

Theorem 4.3 *The mapping $RM_d : H_d \rightarrow \mathcal{P}_d^X$ transforms isomorphically the affine sets $L_q^h(\mu)$ into the Prony leaves $S_q^X(\mu)$, while its inverse \mathcal{V}_d transforms back $S_q^X(\mu)$ into $L_q^h(\mu)$. For each μ the mapping*

$$RM_d : L_q^h(\mu) \rightarrow S_q^X(\mu)$$

provides an isomorphic parametrization of the Prony leaves $S_q^X(\mu)$.

In other words, the Prony leaves $S_q^X(\mu) \subset \mathcal{P}_d^X$ are parametrized by the hyperbolic polynomials $Q \in L_q^h(\mu)$, $Q(z) = z^d + \sigma_1 z^{d-1} + \dots + \sigma_d$, via associating to Q its ordered roots $X = (x_1, \dots, x_d) \in \mathcal{P}_d^X$.

Proof: By Theorem 4.2 the Prony leaf $S_q^X(\mu) \subset \mathcal{P}_d^X$ consists of all $X = (x_1, \dots, x_d) \in \mathcal{P}_d^X$ satisfying equations (4.5). In other words, $S_q^X(\mu)$ consists of all X for which $\mathcal{V}_d(X) \in L_q(\mu)$.

On the other hand, since all the nodes of the signals $F = (A, X) \in \mathcal{P}_d$ are real and pairwise different, the necessary and sufficient condition for σ to have a form $\sigma = \sigma(X)$ is that all the roots of the polynomial $Q(z) = z^d + \sigma_1 z^{d-1} + \dots + \sigma_d$ be real and pairwise different, i.e $Q \in L_q^h(\mu)$. As a conclusion, associating to each $Q \in L_q^h(\mu)$ its ordered roots $X = (x_1, \dots, x_d) \in \mathcal{P}_d^X$ provides the required parametrization of S_q^X . \square

An immediate consequence of Theorem 4.3 is that the Prony leaves $S_q(\mu)$ are smooth algebraic submanifolds in \mathcal{P}_d . We expect that the results of [5, 19–21, 24] will be relevant in further investigation of the geometry and topology of the Prony leaves.

4.3 Prony leaves S_q in the case of two nodes

Here we illustrate the results of Sections 4.1 and 4.2, providing a complete description of the Prony leaves in the case of two nodes, i.e. for $d = 2$.

For $q = 0$ and $\mu = (\mu_0)$ the leaves $S_0(\mu)$ are three-dimensional hyperplanes in $\mathcal{P}_2 \cong \mathbb{R}^4$, defined by the equation $a_1 + a_2 = \mu_0$.

For $q = 1 = d - 1$ and $\mu = (\mu_0, \mu_1)$ the leaves $S_1(\mu)$ are two-dimensional subvarieties in \mathcal{P}_2 , defined by the equations

$$a_1 + a_2 = \mu_0, \quad a_1 x_1 + a_2 x_2 = \mu_1. \quad (4.8)$$

This gives

$$a_1 = \frac{\mu_0 x_2 - \mu_1}{x_2 - x_1}, \quad a_2 = \frac{-\mu_0 x_1 + \mu_1}{x_2 - x_1}, \quad (4.9)$$

which is a special case, for $d = 2$, of expressions (4.3).

Consider now the case $q = 2 = 2d - 2$, and $\mu = (\mu_0, \mu_1, \mu_2)$. Here the leaves $S_2(\mu)$ are (generically) algebraic curves in \mathcal{P}_2 , defined by the equations

$$a_1 + a_2 = \mu_0, \quad a_1 x_1 + a_2 x_2 = \mu_1, \quad a_1 x_1^2 + a_2 x_2^2 = \mu_2, \quad (4.10)$$

For the corresponding curve $S_2^X(\mu)$ in the nodes space $\mathcal{P}_2^X \cong \mathbb{R}^2$ we obtain from Theorem 4.2 the equation $\mu_1 \sigma_1 + \mu_0 \sigma_2 = -\mu_2$, or

$$\mu_0 x_1 x_2 - \mu_1 (x_1 + x_2) + \mu_2 = 0. \quad (4.11)$$

This equation leads to three different possibilities:

1. If $\mu_0 \neq 0$, then the curve $S_2^X(\mu)$ is a hyperbola

$$(x_1 - \frac{\mu_1}{\mu_0})(x_2 - \frac{\mu_1}{\mu_0}) + \frac{\mu_0 \mu_2 - \mu_1^2}{\mu_0^2} = 0, \quad (4.12)$$

which is non-singular for $\mu_0 \mu_2 - \mu_1^2 \neq 0$, and degenerates into two orthogonal coordinate lines, crossing at the diagonal $\{x_1 = x_2\}$, for $\mu_0 \mu_2 - \mu_1^2 = 0$.

2. If $\mu_0 = 0$, but $\mu_1 \neq 0$ then the curve $S_2^X(\mu)$ is a straight line

$$x_1 + x_2 = \mu_2 / \mu_1. \quad (4.13)$$

3. Finally, if $\mu_0 = \mu_1 = 0$, but $\mu_2 \neq 0$ then the curve $S_2^X(\mu)$ is empty, and for $\mu_0 = \mu_1 = \mu_2 = 0$ it coincides with the entire plane \mathcal{P}_2^x . Compare a discussion in the remark after Theorem 4.2.

It is instructive to interpret the cases (1-3) above in terms of the relative position, with respect to the set H_2 of hyperbolic polynomials Q , of the straight line $L_2(\mu)$. This line is defined in the space V_2 of the polynomials $Q(z) = z^2 + \sigma_1 z + \sigma_2$ by system (4.5), i.e. by the equation $\mu_1 \sigma_1 + \mu_0 \sigma_2 = -\mu_2$. Figure 3 illustrates possible positions of the line $L_2(\mu)$ with respect to the set H_2 of hyperbolic polynomials.

The discriminant $\Delta(\sigma_1, \sigma_2) = \sigma_1^2 - 4\sigma_2$ of $Q(z) = z^2 + \sigma_1 z + \sigma_2$ is positive for $Q \in H_2$. Therefore H_2 is the part under the parabola $P = \{\sigma_2 = \frac{1}{4}\sigma_1^2\}$ in V_2 . (Compare Figure 3). The case $\mu_0 \neq 0$ corresponds to the lines $L_2(\mu)$, nonparallel to the σ_2 -axis of V_2 . These lines may cross the parabola P at two points (line l_1 on Figure 3), at one point, if tangent to P (line l_2 on Figure 3), or they may not cross P at all, and then they are entirely contained in H_2 (line l_3 on Figure 3). These cases correspond to $\mu_0 \mu_2 - \mu_1^2 < 0$, $\mu_0 \mu_2 - \mu_1^2 = 0$ and $\mu_0 \mu_2 - \mu_1^2 > 0$, respectively.

For the line $L_2(\mu)$ crossing the parabola P at two points the corresponding hyperbola $S_2^X(\mu)$ crosses the diagonal in the plane \mathcal{P}_2^X , i.e. it contains a collision of the nodes x_1, x_2 (Figure 3, a).

For the line $L_2(\mu)$ tangent to the parabola P , the corresponding hyperbola $S_2^X(\mu)$ degenerates into two orthogonal coordinate lines, crossing at a certain point on the diagonal $\{x_1 = x_2\}$, (Figure 3, b).

For the line $L_2(\mu)$ entirely contained in H_2 the corresponding hyperbola $S_2^X(\mu)$ does not cross the diagonal $\{x_1 = x_2\}$, and so it does not lead to the nodes collision (Figure 3, c).

For $\mu_0 = 0$, but $\mu_1 \neq 0$, the lines $L_2(\mu)$ are parallel to the σ_2 -axis of V_2 , they cross the parabola P at exactly one point (line l_4 on Figure 3). The corresponding curve $S_2^X(\mu)$ is a straight line $x_1 + x_2 = -\frac{\mu_2}{\mu_1}$ (Figure 3, d).

5 Some open questions

The results presented in Section 3 illustrate the role of the Prony leaves in the analysis of the error amplification. The main open problems in the line of this paper concern the structure of the Prony leaves in the areas not covered by the inverse function theorem (Theorem 3.3 above). These areas are collision singularities, on one side, and “escape to infinity” on the other. Both scenarios are frequent in numerical simulations, but we concentrate

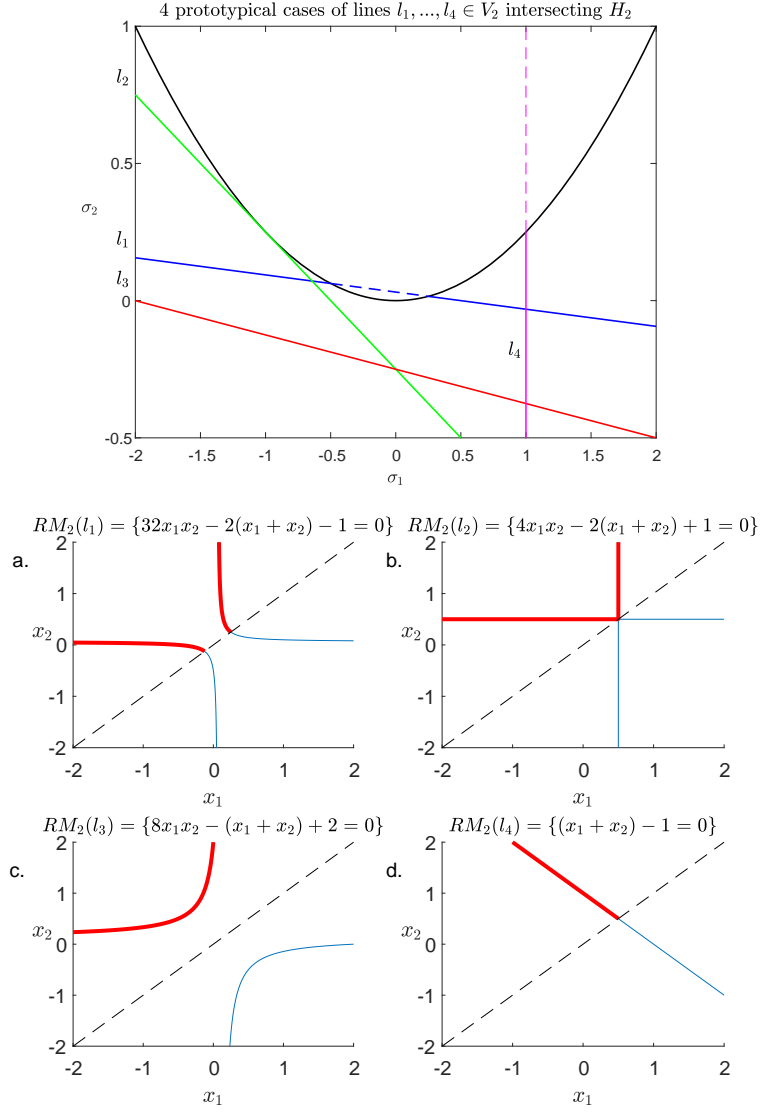


Figure 3: Visualized is the isomorphism $RM_d : H_d \rightarrow \mathcal{P}_d^X$ acting on 4 prototypical lines $l_1, l_2, l_3, l_4 \in V_2$ intersected with H_2 (the open set outside the parabola on the upper figure). In the bottom figure, the highlighted parts in the subplots a,b,c,d are the images, under RM_2 , of l_1, l_2, l_3, l_4 intersected with H_2 , respectively.

on the collision singularities. Let us pose some specific problems in this direction:

1. *Description of the geometry of the nodes x_1, \dots, x_d on the Prony leaves $S_q^x(\mu)$ near the collision singularities.* Presumably, this question can be split into two: investigation of the intersection of the affine varieties $L_q(\mu) \subset V_d$ with the boundary of the hyperbolic set H_d , and investigation of the behavior of the root mapping RM near the boundary of H_d .

We expect that some classical and more recent results on hyperbolic polynomials, Vandermonde varieties (and, more generally, on real roots of polynomials and related topics - see [5, 19–21, 24]) can be relevant. In particular, in [22] some specific straight lines $l \subset V_d$ are described, which are entirely contained in H_d . Can these lines l appear as the lines $L_{2d-2}(\mu)$ for some μ ?

On the other hand, in [24] smooth selections of real roots in families of polynomials Q are described. We expect that this description can be relevant in the study of our families $L_q(\mu)$. “Quantitative” Lojasiewicz-type inequalities may also be useful (see [21] and references therein).

2. *Description of the behavior of the amplitudes a_1, \dots, a_d on the Prony leaves $S_q^X(\mu)$ near the collision singularities.* We expect that this question can be treated via some methods of the classical Moment theory, in combination with the techniques of the “bases of finite differences” developed in [11, 30].

3. *Extending the description of the Prony leaves, and of the error amplification patterns, to multi-cluster nodes configurations.* This is a natural setting in robust inversion of the Prony system. In most practical methods separate clusters are first approximated each by a single node, thus forming a “reduced Prony system”. It is important to estimate the accuracy of such an approximation.

Because of the role of the Prony leaves in the analysis of the error amplification patterns, a natural question is: *To what extent the Prony leaves of the reduced Prony system approximate the leaves of the “true” multi-cluster system?*

References

- [1] Akinshin, A., Batenkov, D. Yomdin, Y. *Accuracy of spike-train Fourier reconstruction for colliding nodes* in Sampling The-

- ory and Applications (SampTA), Pp. 617–621 (IEEE, 2015). doi:10.1109/SAMPTA.2015.7148965
- [2] A.A.Akinshin, V.P.Golubyatnikov, Y.N.Yomdin *Low-dimensional Prony systems* (In Russian), Proc. International Conference “Lomonosov readings in Altai: fundamental problems of science and education”, Barnaul, 20 – 24 October 2015, Altai state university. p. 443 - 450.
 - [3] A.A.Akinshin, G. Goldman, V.P.Golubyatnikov, Y.N.Yomdin, *Accuracy of reconstruction of spike-trains with two near-colliding nodes*, to appear, arXiv:1701.01482.
 - [4] A.A.Akinshin, Batenkov, D, G. Goldman, Y.N.Yomdin, *Error amplification in solving Prony system with near-colliding nodes*, preprint, arXiv:1701.04058.
 - [5] V. Arnold, *Hyperbolic polynomials and Vandermonde mappings*, Funct. Anal. Appl., 20 (1986), pp. 125–127.
 - [6] J. Auton, *Investigation of Procedures for Automatic Resonance Extraction from Noisy Transient Electromagnetics Data*. Volume III. Translation of Prony’s Original Paper and Bibliography of Prony’s Method, Tech. rep., Effects Technology, Santa Barbara, CA, 1981.
 - [7] Azais, Jean-Marc; de Castro, Yohann; Gamboa, Fabrice, *Spike detection from inaccurate samplings*, Appl. Comput. Harmon. Anal. 38 (2015), no. 2, 177195.
 - [8] D. Batenkov, *Accurate solution of near-colliding Prony systems via decimation and homotopy continuation*, to appear in Theoretical Computer Science, 2017. arXiv:1501.00160
 - [9] D. Batenkov, *Stability and super-resolution of generalized spike recovery*, Applied and Computational Harmonic Analysis, Available online 5 October 2016, ISSN 1063-5203, <http://dx.doi.org/10.1016/j.acha.2016.09.004>.
 - [10] D. Batenkov and Y. Yomdin, *On the accuracy of solving confluent Prony systems*, SIAM J.Appl.Math., 73(1):134–154, 2013.

- [11] D. Batenkov and Y. Yomdin, *Geometry and Singularities of the Prony mapping*, Journal of Singularities, 10:1–25, 2014.
- [12] G. Beylkin and L. Monzon, *Approximation by exponential sums revisited*, Appl. Comput. Harmon. Anal. 28, 131–149, 2010.
- [13] Emmanuel J. Candès and Carlos Fernandez-Granda, *Super-Resolution from Noisy Data*, Journal of Fourier Analysis and Applications, 19(6):1229–1254, December 2013.
- [14] Emmanuel J. Candès and Carlos Fernandez-Granda, *Towards a Mathematical Theory of Super-resolution*, Communications on Pure and Applied Mathematics, 67(6):906–956, June 2014.
- [15] L. Demanet and N. Nguyen, *The recoverability limit for superresolution via sparsity*, Preprint, 2014. arXiv:1502.01385.
- [16] Laurent Demanet, Deanna Needell, and Nam Nguyen, *Super-resolution via superset selection and pruning*, In Proceedings of the 10th International Conference on Sampling Theory and Applications (SAMPTA), 2013.
- [17] D.L. Donoho, *Superresolution via sparsity constraints*, SIAM Journal on Mathematical Analysis, 23(5):1309–1331, 1992.
- [18] C. Fernandez-Granda, *Super-resolution of point sources via convex programming*, Inf. Inference 5 (2016), no. 3,
- [19] R. Fröberg, B. Shapiro, *On Vandermonde varieties*, Math. Scand. 119 (2016), no. 1, 73–91.
- [20] V. Kostov, *Topics on hyperbolic polynomials in one variable*, Panoramas et Synthèses [Panoramas and Syntheses], 33, Société Mathématique de France, Paris, 2011. vi+141 pp.
- [21] K. Kurdyka, S. Spodzieja and A. Szlachcińska, *Metric Properties of Semialgebraic Mappings*, Discrete Comput Geom (2016) 55:786–800.
- [22] K. Kurdyka, L. Paunescu, *Nuij type pencils of hyperbolic polynomials*, 2015, arXiv:1504.03665v1.

- [23] V. Morgenshtern, E. Candés, *Super-resolution of positive sources: the discrete setup*, SIAM J. Imaging Sci. 9 (2016), no. 1, 412–444.
- [24] A. Parusinskii, A. Rainer, *Regularity of roots of polynomials*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 16 (2016), no. 2, 481–517.
- [25] T. Peter, G. Plonka, *A generalized Prony method for reconstruction of sparse sums of eigenfunctions of linear operators*, Inverse Problems 29 (2013), 025001.
- [26] T. Peter, D. Potts, and M. Tasche. Nonlinear approximation by sums of exponentials and translates. *SIAM Journal on Scientific Computing*, 33(4), 1920-1947, 2011.
- [27] Gerlind Plonka, M. Wischerhoff, *How many Fourier samples are needed for real function reconstruction?*, J. Appl. Math. Comput. 42, 117–137, 2013.
- [28] Daniel Potts and Manfred Tasche, *Fast ESPRIT algorithms based on partial singular value decompositions*, Appl. Numer. Math. 88 (2015), 31–45.
- [29] R. Prony, *Essai experimental et analytique*, J. Ec. Polytech. (Paris), 2 (1795), pp. 24-76.
- [30] Y. Yomdin, *Singularities in algebraic data acquisition*, Real and complex singularities, London Math. Soc. Lecture Note Ser., 380:378–396, 2010.